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An analytical solution for the nonlinear energy spectrum equation by the decomposition method

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Abstract
We discuss the isotropic turbulence decay and solve the energy density spectrum (EDS) equation considering the inertial transfer energy and viscosity terms, using the Heisenberg parameterization. In the present approach, buoyant and shear terms are neglected and turbulence is assumed to be homogeneous and isotropic. The nonlinear integro-differential equation is solved by Adomian's generic decomposition method, which yields an analytical recursive expression and upon truncation gives an approximate solution. We show the resulting EDS and the time-dependent decay of the intensity of the turbulent kinetic energy. Our results prove consistent the Heisenberg parameterization for the transfer term of the inertial energy. The analytical character of the solution permits a validation of the nonlinear details of the physical model.

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1. Introduction

The planetary boundary layer (PBL) is an inherent complex and heterogeneous system, which is in a permanent transition, due to a variety of external and internal factors. A dominant external influence is the surface heat flux as a consequence of the incident solar radiation. Thus one may define two different regimes for the PBL, the convective regime during the day and the stable one during the night. The first approach for the transition process was introduced by Deardorff [9, 10] using large eddy simulation (LES), considering experimental data from the Wangara measurements with the goal to characterize stationary turbulence. Nieuwstadt and Brost [20] analysed the decay of turbulence in the convective boundary layer (CBL) and taking into account LES data. In their approach the authors used an instantaneous cut for the heat flux.

Only after a decade this abrupt day–night transition was improved by a gradual heat flux change with time [25]. Since then a variety of works discussed this issue using LES as for instance in [7, 13, 22, 23]. In the latter work turbulent shear effects during the decay in the CBL were included. A next step was done examining humidity effects close to the terrestrial surface during the day–night transition from experimental data [1, 2]. Numerical investigations of the decay in the CBL gave rise to several observational projects, see, for instance, [12, 17, 19]. Acevedo and Fitzjarrald [1, 2] report on the occurrence of specific humidity during the day–night transition and a temperature drop at the surface accompanied by a sudden decrease in the wind velocity.

Literature is scarce with respect to theoretical treatments of the transition in the CBL. Goulart and coworkers [14, 15] proposed a model for the decay of the energy density spectrum (EDS) in the CBL starting from a spectral energy balance equation, neglecting kinetic energy production by mechanical effects, which gave comparable results to LES. A remarkable experimental effort [6] was performed during the eclipse on 11 August 1999, where the experimental team analysed the EDS decay and found the transition comparable to the changes from day to night. The analyses of the transition periods from spectral models opens pathways for approaches using well-established conservation laws, energy conservation in this case. The efficiency of the model is linked to the significance of the parameter inference of the unknown terms in the energy conservation equation as well as the quality of the mathematical method employed to solve the resulting equation.

In the work of Goulart *et al* [15] the starting point of this study was the balance equation for the EDS, in which the parameterization of the inertial energy transfer term was expressed as a superposition of the time variation of the velocity correlation tensor and the source for convective energy. On the other hand, the buoyancy term in this equation was represented in a factorized form [24] in which the initial heat flux and the function controlling its temporal decrease were described separately. Employing these parameterizations in the equation for EDS, applying the Laplace transform and numerically inverting the transformed spectrum function, a solution representing the decaying convective 3D spectrum was obtained. However, the complexity originating from the aforementioned solution method imposes a severe limitation in distinct applications that consider decaying turbulent properties in CBL. As a shortcoming it is not possible to derive eddy diffusivities for the decaying turbulence in the CBL. On the other hand, differently to the solution method proposed by Goulart *et al* [15] and disregarding the buoyancy term in the equation describing the evolution of the spectrum function, Goulart *et al* [14] derived decaying convective eddy diffusivities. Though, in this approach the inertial energy transfer term has been parameterized following an idea suggested by Pao [21], which turns a nonlinear turbulent equation into a linear one. In order to overcome the shortcomings of [14, 15], and explore especially the nonlinearity, the present study focuses on a new parameterization for the inertial energy transfer term in the balance equation for the EDS. To this end, the inertial term is parameterized using Heisenberg's turbulent spectral transfer theory. As a consequence, this procedure retains the inherent nonlinear character associated with the turbulence problem and yields an integro-differential equation that provides an analytical solution to the decaying 3D energy spectrum. Thus, the major advantage associated with Heisenberg's parameterization is due to the fact that the present approach generates a solution avoiding the usual linearization procedure in the turbulence problem, although its character is intrinsically nonlinear.

2. Dynamical energy density spectrum

In order to model the isotropic turbulence decay we solve the EDS equation considering the inertial transfer energy and viscosity terms. Here we parameterize the inertial transfer energy by the Heisenberg parameterization. This procedure yields a nonlinear integro-differential equation. Since the turbulence decay is still poorly understood, we reason that an analytical solution shall open pathways to understand at least some details of the phenomenon. Hence we solve this equation by the generic decomposition method proposed by Adomian [3–5], which solves analytically nonlinear problems by a recursive procedure. Due to its analytical character of the solution we expect to make progress in the issue of the decay of the energy spectrum.

In the decaying CBL, to a first approximation, buoyant and shear terms can be disregarded and turbulence can be assumed to be homogeneous and isotropic. Consequently, the following three-dimensional EDS equation reads [16, 26]

$$\frac{\partial E(k,t)}{\partial t} = W(k,t) - 2\nu k^2 E(k,t), \tag{1}$$

where t is time, k is the wave number, $E(k,t)$ is the 3D EDS, $W(k,t)$ is the inertial transport term and ν is the kinematic viscosity.

2.1. Heisenberg parameterization

A turbulent flow contains eddies of different sizes or equivalently different wavelengths. The small eddies are subject to the stress generated by larger eddies. This field increases the vorticity of small eddies and, consequently, their kinetic energy. Thus, turbulent kinetic energy is successively transferred from larger eddies to smaller and smaller eddies until the Kolmogorov micro-scale is reached, where the energy is dissipated as heat. This process is represented by the term $W(k,t)$ of equation (1) and was parameterized, according to Heisenberg, for a turbulent isotropic flow on the basis of dimensional analysis, as follows:

$$W(k,t) = -2\nu_t k^2 E(k,t), \tag{2}$$

where

$$\nu_t = C_H \int_k^\infty \sqrt{\frac{E(k',t)}{k'^3}}\,\mathrm{d}k' \tag{3}$$

is the eddy viscosity, with C_H the Heisenberg constant. Upon substituting equations (2) and (3) into equation (1), one obtains

$$\frac{\partial E(k,t)}{\partial t} + 2C_H k^2 E(k,t)\int_k^\infty \sqrt{\frac{E(k',t)}{k'^3}}\,\mathrm{d}k' + 2\nu k^2 E(k,t) = 0. \tag{4}$$

Considering the following dimensionless parameters, in which w_* is the convective velocity scale, z_i is the CBL height and Ψ_ϵ is the nondimensional molecular dissipation rate function,

$$t_* = \frac{w_* t}{z_i} \qquad R_e = \frac{w_* z_i}{\nu} \qquad \Psi_\epsilon = \frac{\epsilon z_i}{w_*^3}, \tag{5}$$

equation (4) becomes

$$\frac{\partial E(\tilde{k},t_*)}{\partial t_*} + \frac{2C_H}{w_* z_i}\tilde{k}^2 E(\tilde{k},t_*)\int_{\tilde{k}}^\infty \sqrt{\frac{E(\tilde{k}',t_*)}{\tilde{k}'^3}}\,\mathrm{d}\tilde{k}' + \frac{2}{R_e}\tilde{k}^2 E(\tilde{k},t_*) = 0 \tag{6}$$

where $\tilde{k} = kz_i$. Using the new definitions allows us to cast equation (6) into a simpler form

$$\frac{\partial E(\tilde{k},t_*)}{\partial t_*} + \tilde{k}^2 F(E(\tilde{k},t_*)) + \frac{2}{R_e}\tilde{k}^2 E(\tilde{k},t_*) = 0, \tag{7}$$

where

$$F(E(\tilde{k}, t_*)) = \frac{2C_H}{w_* z_i} E(\tilde{k}, t_*) \int_{\tilde{k}}^{\infty} \sqrt{\frac{E(\tilde{k}', t_*)}{\tilde{k}'^3}}\, \mathrm{d}\tilde{k}'. \tag{8}$$

2.2. Adomian decomposition

Upon application of the decomposition method [3–5] in equation (7), the method decomposes the linear term into an infinite sum of components

$$E(\tilde{k}, t_*) = \sum_{n=0}^{N} u_n(\tilde{k}, t_*), \tag{9}$$

and the nonlinear term may be identified by the decomposition series

$$F(E(\tilde{k}, t_*)) = \sum_{n=0}^{N} A_n(E(\tilde{k}, t_*)), \tag{10}$$

where A_n are the so-called Adomian polynomials. Adomian [3–5] introduced formulae that can generate polynomials for all forms of nonlinearity. In the following we show the form of the first three polynomials:

$$\begin{aligned} A_0 &= F(u_0) \\ A_1 &= u_1 F'(u_0) \\ A_2 &= u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0) \\ &\cdots. \end{aligned} \tag{11}$$

It is noteworthy that A_0 depends only on u_0, A_1 depends only on u_0 and u_1, and A_2 depends on u_0, u_1 and u_2. In order to solve equation (7) with Adomian's method we substitute (9) and (10) into (7),

$$\frac{\partial}{\partial t_*}\left(\sum_{n=0}^{N} u_n(\tilde{k}, t_*)\right) + \left(\sum_{n=0}^{N} A_n(E(\tilde{k}, t_*))\right)\tilde{k}^2 + \frac{2}{R_e}\tilde{k}^2\left(\sum_{n=0}^{N} u_n(\tilde{k}, t_*)\right) = 0. \tag{12}$$

We can write the equation above as a system recursive equations,

$$\begin{aligned} \frac{\partial u_0}{\partial t_*} + \frac{2}{R_e}\tilde{k}^2 u_0 &= 0 \\ \frac{\partial u_1}{\partial t_*} + \frac{2}{R_e}\tilde{k}^2 u_1 &= -\tilde{k}^2 A_0 \\ \frac{\partial u_2}{\partial t_*} + \frac{2}{R_e}\tilde{k}^2 u_2 &= -\tilde{k}^2 A_1. \end{aligned}$$

The general equation for u is

$$\frac{\partial u_n}{\partial t_*} + \frac{2}{R_e}\tilde{k}^2 u_n = -\tilde{k}^2 A_{n-1}, \tag{13}$$

and the analytical solution is given by

$$u_n = E_0 \mathrm{e}^{-\frac{2}{R_e}\tilde{k}^2 t_*} + \mathrm{e}^{-\frac{2}{R_e}\tilde{k}^2 t_*} \int_0^{t_*} A_{n-1} \mathrm{e}^{-\frac{2}{R_e}\tilde{k}^2 t_*'}\, \mathrm{d}t_*', \tag{14}$$

where E_0 is the initial spectrum of the CBL. For the spectral function given by equation (10) we will consider $n = 3$, the same holds for equation (14).

2.3. The energy density spectrum

Here non-isotropic initial turbulence is considered, and consequently, the formulation proposed by Kristensen and coworkers [18] may be employed to determine the initial 3D spectrum of the CBL. This formulation allows determining the 3D spectrum of a homogeneous turbulent flow from a known 1D spectrum,

$$E_0(k,z) = k^3 \frac{\mathrm{d}}{\mathrm{d}k}\frac{1}{k}\frac{\mathrm{d}S_u(k)}{\mathrm{d}k} + 12 A_i m_i B_i^{\frac{17}{6}} k^4 \sum_{n=0}^{3} C_n \int_{T_{1i}}^{\infty} \frac{X_i^{3n-12}}{\left(X_i^3-1\right)^5}\,\mathrm{d}X_i - \frac{84}{9} A_i m_i B_i^{\frac{4}{3}} \sum_{n=0}^{3} C_n \int_1^{T_{2i}} \frac{X_i^{3n-12}}{\left(X_i^3-1\right)^{n-5}}\,\mathrm{d}X_i \tag{15}$$

with $i = u, v, w$ and

$$T_{1i} = \left(1 + \frac{1}{\sqrt{B_i s}}\right)^{\frac{1}{3}} \qquad T_{2i} = \left(1 + \sqrt{B_i s}\right)^{\frac{1}{3}}$$
$$A_i = a_i b_i^{-\frac{5}{6}} \qquad B_i = b_i^{-2}$$
$$m_u = 2 \qquad m_v = m_w = 1$$
$$C_0 = -\frac{55}{27} \qquad C_1 = \frac{70}{9} \qquad C_2 = \frac{725}{72} \qquad C_3 = \frac{935}{216}.$$

According to Degrazia and Anfossi [11] the initial 1D spectrum can be written as

$$S_i(k,0) = \frac{a_i}{(1+b_i k)^{\frac{5}{3}}}, \tag{16}$$

where

$$a_i = \frac{0.98}{2\pi} c_i \left(\frac{z}{z_i}\right)^{\frac{5}{3}} z_i \Psi_\epsilon^{\frac{2}{3}} w_*^2 \left(\left(f_m^*\right)_i^c\right)^{-\frac{5}{3}} \qquad \text{and} \qquad b_i = \frac{1.5}{2\pi} z \frac{1}{\left(f_m^*\right)_i^c}$$

with (see [8])

$$c_i = \alpha_i (0.5 \pm 0.05)(2\pi\kappa)^{-\frac{2}{3}} \qquad \alpha_i = 1, \frac{4}{3}, \frac{4}{3}$$

and

$$w_* = (u_*)_0 \left(\frac{z_i}{\kappa L}\right)^{\frac{1}{3}}, \qquad \left(f_m^*\right)_i^c = \frac{z}{G_i z_i}, \qquad G_u = G_v = 1.5,$$
$$G_w = 1.8\left(1 - \mathrm{e}^{-\frac{4z}{z_i}} - 0.0003\mathrm{e}^{-\frac{8z}{z_i}}\right).$$

Equation (16) is an empirical expression that represents the observed one-dimensional spectra for a stationary CBL. This expression is used to construct the initial condition (equation (15)) for an isotropic model describing the decaying of the anisotropic turbulence energy spectrum. The EDS calculated from equations (9), (14)–(16) can be integrated to get the kinetic total energy, that is

$$K = \int_0^{\infty} E(k,t)\,\mathrm{d}k. \tag{17}$$

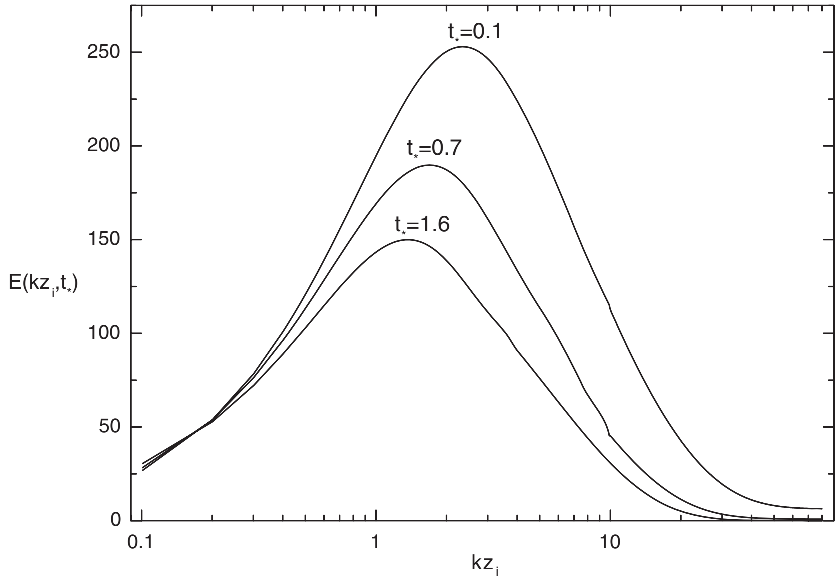

Figure 1. Energy density spectrum in the decaying CBL.

3. Results

Figure 1 shows the EDS calculated from equations (9) (with $N = 3$), (14)–(16). One observes that the intensity of the turbulent kinetic energy diminishes with time and the maximum of the distribution moves to smaller wave numbers. This behaviour is expected from the physical point of view, because small eddies (large wave numbers) decay faster than larger ones, so that with increasing time eddies with increasingly smaller wave numbers survive, manifest in the temporal displacement of the maximum in the spectrum.

Figure 2 (solid line) shows the kinetic energy calculated from the model presented in this work with Heisenberg's parameterization for the inertial energy transfer term (equations (17), (9), (14)–(16)). Results obtained using Pao's parameterization [14] and from the LES data (compiled from [20]) are included into the figure (dotted line and point set, respectively). One observes that the turbulent kinetic energy determined by employing the Heisenberg parameterization from equations (17), (9), (14)–(16) maintains approximately a constant value until $t_* \approx 1$ comparable to the results from [20], which in turn proves plausible the proposed parameterization of the energy transfer (equations (2) and (3)). Recalling that these results are obtained only considering the inertial energy transfer and energy dissipation through viscosity effects, both decreasing the energy. An inadequate parameterization of the kinetic energy transfer by inertial effects should manifest itself in the nonlinear contribution with the result that eddies had a faster decrease in the turbulent kinetic energy, especially in comparison to the simulation of Nieuwstadt and Brost [20].

Particularly, the results obtained employing the Heisenberg parameterization agree fairly well with the LES data. Differently, Pao's parameterization does not reproduces adequately the results generated from the LES data. This disparity between Heisenberg and Pao parameterizations can be explained by the fact that the inertial energy transfer is a typically nonlinear phenomenon and any procedure of linearization to reproduce this term represents

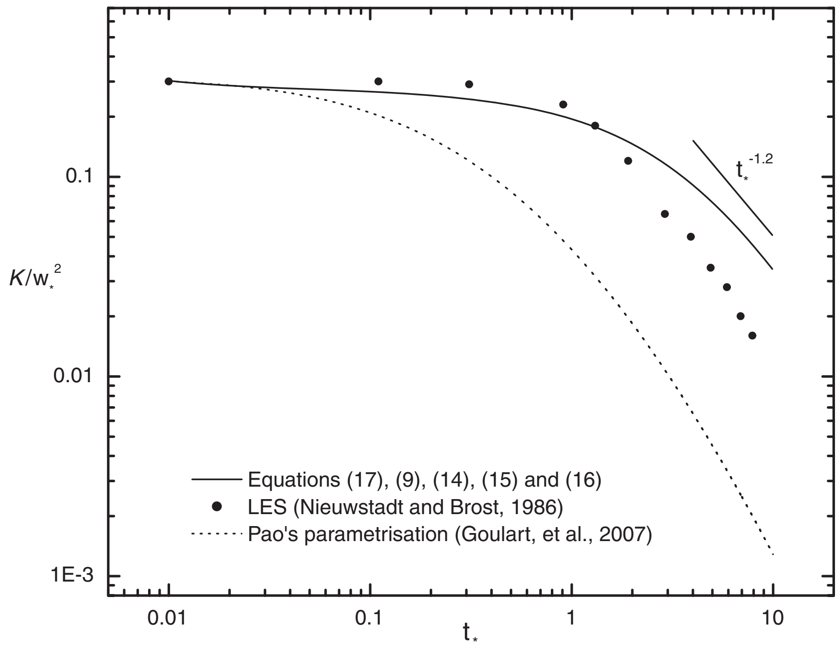

Figure 2. Temporal evolution of the kinetic energy decay of the CBL.

a crude approximation. In this context the nonlinear equation (4) derived, assuming the Heisenberg hypothesis is solved in this work without linearization.

4. Conclusion

The present work is, to the best of our knowledge, the first successful approach to derive an analytical expression for the three-dimensional turbulent kinetic energy spectrum, neglecting a mechanical energy contribution and considering the turbulence homogeneous and isotropic. The fact that we have taken into account the nonlinearity and have found a closed form solution following Adomian's prescription represents the principal progress in the question of turbulence in the day–night transition of the CBL. From a formal point of view the new solution is manifest exact since in principle no approximation is made along its derivation, except for the truncation of the decomposition series. This reinforces our conclusions from the comparison of the solution by Adomian's method to the solution with Pao's linearized parameterization. The discrepancy (shown in figure 2) shall be mainly due to the physical difference by the nonlinear contributions. In this sense, our solution does not only provide a more adequate description of turbulence decay but further sheds light on the influence of the linearization in the inertial transport term. Numerical large eddy simulations and our findings are roughly in agreement, whereas the missing nonlinear effects seem to destabilize turbulence which leads to a precocious decay, approximately one order of magnitude earlier than our result or the LES data. Encouraged by the good results we focus our future attention to the task of application of this methodology to solve the energy spectral equation considering mechanical and convective turbulent energy.

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